

Lecture 46

Divergence: Let $\vec{V}$ be any given differentiable vector function. Then the divergence of $\vec{V}$ is given by

$$\begin{aligned}\vec{\nabla} \cdot \vec{V} &= \text{div}(\vec{V}) = \text{div} \vec{V} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (V_1, V_2, V_3) \\ &= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.\end{aligned}$$

Curl of a vector function: Let $\vec{V}$ be any given differentiable vector function. Then the curl of $\vec{V}$ is given by

$$\begin{aligned}\vec{\nabla} \times \vec{V} &= \text{curl}(\vec{V}) = \text{curl} \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \\ &= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) \hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \hat{k}.\end{aligned}$$

Irrotational vector: A vector function $\vec{V}$ is said to be irrotational if $\text{curl} \vec{V} = 0$. In other words, there exists a scalar function φ such that $\vec{V} = -\vec{\nabla} \varphi$.

Solenoidal vector: A vector function $\vec{V}$ is said to be solenoidal if $\text{div} \vec{V} = 0$.

Laplacian of a scalar: The Laplacian operator ∇^2 is defined as

$$\Delta := \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Laplace equation: $\text{div}(\vec{\nabla} u) = \nabla^2 u = 0$.

Example 1. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\begin{aligned}\text{div} \vec{r} &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3. \\ \text{curl} \vec{r} &= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) \hat{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) \hat{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \hat{k} = \vec{0}.\end{aligned}$$

Example 2. Let $\vec{V} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$. Then $\text{div} \vec{V} = 1 + 1 + a$. Therefore, $\vec{V}$ is solenoidal if $a + 2 = 0$, i.e., $a = -2$.

Example 3. Let $\vec{V} = (\sin y + z)\hat{i} + (x \cos y - z)\hat{j} + (x - y)\hat{k}$. Then

$$\begin{aligned}\text{curl} \vec{V} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x \cos y - z & x - y \end{vmatrix} \\ &= (-1 + 1)\hat{i} + (1 - 1)\hat{j} + (\cos y - \cos y)\hat{k} \\ &= \vec{0}.\end{aligned}$$

Therefore, $\vec{V}$ is an irrotational vector.

Vector identities in terms of div & curl : Let $\vec{V}$ be a vector function and φ be a scalar function.

- (i) $\text{div } \varphi \vec{V} = \vec{\nabla} \varphi \cdot \vec{V} + \varphi \text{div } \vec{V}.$
- (ii) $\text{curl } \varphi \vec{V} = \vec{\nabla} \varphi \times \vec{V} + \varphi \text{curl } \vec{V}.$
- (iii) $\vec{\nabla}(\vec{f} \cdot \vec{g}) = (\vec{g} \cdot \vec{\nabla}) \vec{f} + (\vec{f} \cdot \vec{\nabla}) \vec{g} + \vec{g} \times \text{curl } \vec{f} + \vec{f} \times \text{curl } \vec{g}.$

Property: For any vector function $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$, $\text{div}(\text{curl } \vec{V}) = 0.$

Proof.

$$\begin{aligned} \text{div}(\text{curl } \vec{V}) &= \vec{\nabla} \cdot \left(\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) \hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \hat{k} \right) \\ &= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y} \\ &= 0. \end{aligned}$$

□

Lecture 47

Application to Mechanics

Curve in space: A curve is an aggregate of points where coordinates are function of a single variable. Thus the equations

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

represents a curve in space and the variable “ t ” is called the parameter. For each value of $t \in \mathbb{R}$ (i.e., $a \leq t \leq b \in \mathbb{R}$), there corresponds a definite point $P(x, y, z)$ of the curve.

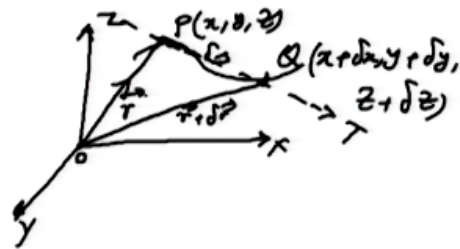
Example 4.

- (i) In xy -plane: $x^2 + y^2 = 25$ is a circle. $x = 5 \cos t$, and $y = 5 \sin t$ where $0 \leq t \leq 2\pi$.
- (ii) In xyz -plane: $x^2 + y^2 + z^2 = 25$ is a sphere. $x = 5 \cos \theta \cos \phi$, $y = 5 \cos \theta \sin \phi$, and $z = 5 \sin \theta$ where $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$.

Let $\vec{r} = \vec{f}(t)$ represent the equation of a curve.
Now,

$$\begin{aligned} \vec{r} &= x \hat{i} + y \hat{j} + z \hat{k} \\ &= x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}, \quad t \in \mathbb{R}. \end{aligned}$$

From here, we take $x = x(t)$, $y = y(t)$, and $z = z(t)$. The tangent line PT at a point P of a curve is the limiting position of the secant PQ joining P to a



neighboring point Q , i.e., when Q approaches P along the curve.

$$\begin{aligned}\vec{PQ} &= \vec{OQ} - \vec{OP} = \vec{r} + \delta\vec{r} - \vec{r} = \vec{f}(t + \delta t) - \vec{f}(t) \\ \lim_{\delta t \rightarrow 0} \frac{\vec{PQ}}{\delta t} &= \lim_{\delta t \rightarrow 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t} \\ \frac{d\vec{r}}{dt} &= \vec{f}'(t)\end{aligned}$$

That is, $\frac{d\vec{r}}{dt} = \vec{f}'(t)$ is parallel to the tangent PT of the curve $\vec{r} = \vec{f}(t)$ at P . Therefore,

$$\begin{aligned}\vec{R} &= \vec{r} + \lambda \frac{d\vec{r}}{dt} \\ \vec{R} - \vec{r} &= \lambda \frac{d\vec{r}}{dt}, \quad (\text{Equation of the tangent line}) \\ \text{or, } (\vec{R} - \vec{r}) \times \frac{d\vec{r}}{dt} &= \vec{0}.\end{aligned}$$

where λ is any arbitrary constant, $\vec{r}$ is the position vector of the P and $\vec{R}$ is the position of any point on the tangent line.

Example 5. Find the equation of a tangent line to the space curve $x = t$, $y = t^2$, and $z = \frac{2}{3}t^3$ at $t = 1$.

Solution: We can write $\vec{r}(t) = t\hat{i} + t^2\hat{j} + \frac{2}{3}t^3\hat{k}$. Therefore, the position of the point at $t = 1$ is $\vec{P} = \hat{i} + \hat{j} + \frac{2}{3}\hat{k}$.

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \hat{i} + 2t\hat{j} + 2t^2\hat{k} \\ \left. \frac{d\vec{r}}{dt} \right|_{t=1} &= \hat{i} + 2\hat{j} + 2\hat{k}.\end{aligned}$$

Hence, the equation of the tangent line

$$\begin{aligned}((X\hat{i} + Y\hat{j} + Z\hat{k}) - (\hat{i} + \hat{j} + \frac{2}{3}\hat{k})) \times (\hat{i} + 2\hat{j} + 2\hat{k}) &= \vec{0} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X-1 & Y-1 & Z-\frac{2}{3} \\ 1 & 2 & 2 \end{vmatrix} &= \vec{0} \\ (2Y - 2Z - \frac{2}{3})\hat{i} + (Z - 2X + \frac{4}{3})\hat{j} + (2X - Y - 1)\hat{k} &= \vec{0}\end{aligned}$$

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Unit tangent vector: Let s be the arc length and $\hat{t}$ be the unit tangent vector. Then $\frac{d\vec{r}}{ds} = \hat{t}$ and the equation of the tangent line will be

$$(\vec{R} - \vec{r}) \times \hat{t} = \vec{0}.$$

Unit principle normal vector: As the length of $\hat{t}$ is constant, its derivative $\frac{d\hat{t}}{ds}$, if not zero, must be perpendicular to $\hat{t}$. Therefore, it will be normal to the curve at the point P . A directed line through P in the direction of $\frac{d\hat{t}}{ds}$ is called principle normal vector to the curve

C at the point P . If $\hat{n}$ is the unit vector in the direction of principle normal vector, then we may write

$$\frac{d\hat{t}}{ds} = \kappa\hat{n},$$

where κ is a non-negative scalar and $\hat{n}$ is the unit principle normal. κ is the called curvature of the curve at the point P .

Unit binormal vector: We introduce an another unit normal vector $\hat{b}$ defined by $\hat{b} = \hat{t} \times \hat{n}$, where $\hat{b}$, $\hat{t}$, and $\hat{n}$ forms a right-handed system of orthogonal vectors. A directed line through P in the direction of $\hat{b}$ is called binormal to the curve at the point P . We may call $\hat{b}$ as unit binormal at P .

Torsion: We have defined $\hat{b} = \hat{t} \times \hat{n}$ and since $\hat{b}$ is a unit vector, $\frac{d\hat{b}}{ds}$, if not zero, must be perpendicular to $\hat{b}$, then

$$\begin{aligned}\hat{b} &= \hat{t} \times \hat{n} \\ \frac{d\hat{b}}{ds} &= \frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \\ &= \kappa\hat{n} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \\ \frac{d\hat{b}}{ds} &= \hat{t} \times \frac{d\hat{n}}{ds}.\end{aligned}$$

Then $\frac{d\hat{b}}{ds}$ is perpendicular to $\hat{t}$. But it is also perpendicular to $\hat{b}$. Therefore, it must be parallel to $\hat{n}$. Then

$$\frac{d\hat{b}}{ds} = -\tau\hat{n},$$

where τ is a scalar, is called torsion of the curve at the point P . The minus sign is taken because when τ is positive, $\frac{d\hat{b}}{ds}$ has the direction of $-\hat{n}$ and P moves along with positive direction.

Again,

$$\begin{aligned}\hat{n} &= \hat{b} \times \hat{t} \\ \frac{d\hat{n}}{ds} &= \frac{d\hat{b}}{ds} \times \hat{t} + \hat{b} \times \frac{d\hat{t}}{ds} \\ &= -\tau\hat{n} \times \hat{t} + \hat{b} \times \kappa\hat{n} \\ &= \tau\hat{b} - \kappa\hat{t}.\end{aligned}$$

Summarizing the relations we have

$$(i) \quad \hat{b} = \hat{t} \times \hat{n}, \quad \hat{n} = \hat{b} \times \hat{t}, \quad \hat{t} = \hat{n} \times \hat{b}.$$

$$(ii) \quad \textbf{Serret-Frenet Formula:} \quad \frac{d\hat{t}}{ds} = \kappa\hat{n}, \quad \frac{d\hat{b}}{ds} = -\tau\hat{n}, \quad \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}.$$

Let $\vec{r} = \vec{f}(t)$ be a vector function.

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \dot{\vec{r}} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \hat{t}\dot{s} \\ |\dot{\vec{r}}| &= |\hat{t}||\dot{s}| = |\dot{s}| \\ \ddot{\vec{r}} &= \frac{d\dot{\vec{r}}}{ds} \dot{s}^2 + \hat{t}\ddot{s} = \kappa\hat{n}\dot{s}^2 + \hat{t}\ddot{s} \\ \ddot{\vec{r}} &= (\ddot{s} - \kappa^2\dot{s})\hat{t} + (3\dot{s}\ddot{s}\kappa + \dot{s}^2\kappa)\hat{n} + \dot{s}^3\kappa\tau\hat{b}\end{aligned}$$

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The three vectors $\hat{t}$, $\hat{n}$, and $\hat{b}$ are perpendicular to each other. The plane corresponds to each pair forms three mutually perpendicular planes.

- (i) **Osculating Plane:** The osculating plane to a curve at a point P is the plane containing the tangent and principle normal at P . The equation of osculating plane is

$$(\vec{R} - \vec{r}) \cdot \hat{b} = 0,$$

where $\vec{R}$ is the position of any point on the plane, and $\vec{r}$ is the position vector of the point P .

- (ii) **Normal plane:** It is a plane to a curve at a point P containing $\hat{n}$ and $\hat{b}$, and perpendicular to $\hat{t}$. The equation of the normal plane is

$$(\vec{R} - \vec{r}) \cdot \hat{t} = 0.$$

- (iii) **Rectifying plane:** It is a plane to a curve at a point P containing $\hat{b}$ and $\hat{t}$, and perpendicular to $\hat{n}$. The equation of the rectifying plane is

$$(\vec{R} - \vec{r}) \cdot \hat{n} = 0.$$

Example 6. Find $\hat{t}$, $\hat{b}$, $\hat{n}$, κ and τ for the curve $x = 2t$, $y = t^2$ and $z = \frac{1}{3}t^3$ at $t = 1$. Also find the equation of osculating plane, normal plane, and rectifying plane.

Solution: The given equation of the curve can be put into the vector form as

$$\begin{aligned}\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ &= 2t\hat{i} + t^2\hat{j} + \frac{1}{3}t^3\hat{k}\end{aligned}$$

$$\frac{d\vec{r}}{dt} = 2\hat{i} + 2t\hat{j} + t^2\hat{k}, \quad \frac{d^2\vec{r}}{dt^2} = 2\hat{j} + 2t\hat{k}, \quad \frac{d^3\vec{r}}{dt^3} = 2\hat{k}.$$

At $t = 1$:

$$\vec{r} = 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}, \quad \frac{d\vec{r}}{dt} = 2\hat{i} + 2\hat{j} + \hat{k}, \quad \frac{d^2\vec{r}}{dt^2} = 2\hat{j} + 2\hat{k}, \quad \frac{d^3\vec{r}}{dt^3} = 2\hat{k}.$$

Also, $\dot{\vec{r}} \times \ddot{\vec{r}} = (2\hat{i} + 2\hat{j} + \hat{k}) \times (2\hat{j} + 2\hat{k}) = 2\hat{i} - 4\hat{j} + 4\hat{k}$, and $(\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \ddot{\vec{r}} = 8$.

$$(i) \quad \hat{t} = \frac{d\vec{r}}{dt} / \left| \frac{d\vec{r}}{dt} \right| = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{2^2 + 2^2 + 1}} = \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k}).$$

(ii) As $\ddot{\vec{r}} = \kappa \hat{n} \dot{s}^2 + \hat{t} \ddot{s}$, and $\ddot{\vec{r}} = (\ddot{s} - \kappa^2 \dot{s})\hat{t} + (3\dot{s}\ddot{s}\kappa + \dot{s}^2\kappa)\hat{n} + \dot{s}^3\kappa\tau\hat{b}$, we obtain

$$\begin{aligned}\dot{\vec{r}} \times \ddot{\vec{r}} &= \hat{t} \dot{s} \times (\kappa \hat{n} \dot{s}^2 + \hat{t} \ddot{s}) = \kappa \hat{b} \dot{s}^3 \\ \kappa \hat{b} &= \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\dot{s}^3}\end{aligned}$$

The binormal vector can be chosen as $\hat{b} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$ such that $\hat{b} \cdot \hat{t} = 0$.

$$(iii) \quad \hat{n} = \hat{b} \times \hat{t} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) \times \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k}) = \frac{1}{3}(-2\hat{i} + \hat{j} + 2\hat{k}).$$

(iv)

$$\begin{aligned}\kappa \hat{b} &= \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\dot{s}^3} \\ \kappa &= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{s}|^3} = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \\ &= \frac{|2\hat{i} - 4\hat{j} + 4\hat{k}|}{|2\hat{i} + 2\hat{j} + \hat{k}|^3} = \frac{2}{9}.\end{aligned}$$

$$(v) \quad \tau = \frac{(\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \dddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} = \frac{8}{|2\hat{i} - 4\hat{j} + 4\hat{k}|^2} = \frac{2}{9}.$$

Let $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position of any point on the plane.

(i) The equation of osculating plane:

$$\begin{aligned}(\vec{R} - \vec{r}) \cdot \hat{b} &= 0 \\ ((X\hat{i} + Y\hat{j} + Z\hat{k}) - 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}) \cdot \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) &= 0 \\ (X - 2) - 2(Y - 1) + 2(Z - \frac{1}{3}) &= 0 \\ X - 2Y + 2Z &= \frac{2}{3}\end{aligned}$$

(ii) The equation of normal plane:

$$\begin{aligned}(\vec{R} - \vec{r}) \cdot \hat{t} &= 0 \\ ((X\hat{i} + Y\hat{j} + Z\hat{k}) - 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}) \cdot \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k}) &= 0 \\ 2(X - 2) + 2(Y - 1) + (Z - \frac{1}{3}) &= 0 \\ 2X + 2Y + Z &= \frac{19}{3}\end{aligned}$$

(iii) The equation of rectifying plane:

$$\begin{aligned}(\vec{R} - \vec{r}) \cdot \hat{n} &= 0 \\ ((X\hat{i} + Y\hat{j} + Z\hat{k}) - 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}) \cdot \frac{1}{3}(-2\hat{i} + \hat{j} + 2\hat{k}) &= 0 \\ -2(X - 2) + (Y - 1) + 2(Z - \frac{1}{3}) &= 0 \\ 2X - Y - 2Z &= \frac{7}{3}\end{aligned}$$

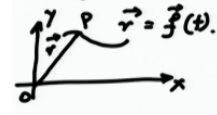
Lecture 50

Application of vector calculus in Mechanics

Velocity: The velocity of a particle relative to a suitable frame of reference is the time rate of change of the position vector $\vec{r}$ of the particle relative to the given frame of reference.

Let $\vec{OP} = \vec{r}$. At any time interval Δt , the increment in $\vec{r}$ be $\Delta \vec{r}$, then $\frac{\Delta \vec{r}}{\Delta t}$ is the average velocity of P relative to O during the interval Δt . Therefore, the velocity of the particle P at time t is given by

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \vec{r}_t.$$



Example 7. Let $\vec{r} = e^t \hat{i} + t^2 \hat{j} + \sin t \hat{k}$ be the position of a particle at time t . Then velocity $\vec{v} = \frac{d\vec{r}}{dt} = e^t \hat{i} + 2t \hat{j} + \cos t \hat{k}$.

At time $t = 1$,

the velocity of the particle: $\left. \frac{d\vec{r}}{dt} \right|_{t=1} = e \hat{i} + 2 \hat{j} + \cos 1 \hat{k}$.

magnitude of the velocity: $|\vec{v}| = \sqrt{e^2 + 4 + \cos^2 t}$.

Acceleration: It is a time rate of change of velocity. If $\vec{OP} = \vec{r}$ is the position of a particle then acceleration of the particle at time t is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left(\frac{d\vec{r}}{dt} \right) = \frac{d^2 \vec{r}}{dt^2}.$$

Example 8. Let $\vec{r}(t) = (t^2 - 1) \hat{i} + \sin^2 t \hat{j} + t^3 \hat{k}$ be the position of a particle at time t .

Velocity: $\vec{v} = \dot{\vec{r}} = 2t \hat{i} + 2 \sin t \cos t \hat{j} + 3t^2 \hat{k}$

Acceleration: $\vec{a} = \dot{\vec{v}} = 2 \hat{i} + 2 \cos 2t \hat{j} + 6t \hat{k}$

Equation of motion for a particle

Momentum: By momentum $\vec{p}$ of a moving particle P at any time t , we mean the vector $m\vec{v}$, where m is the mass of the particle and $\vec{v}$ is its velocity, i.e.,

$$\vec{p} = m\vec{v}.$$

Moment of momentum: If $\vec{p}$ is the linear momentum of a particle P at any instant of time t , then $\vec{OP} \times \vec{p} = \vec{H}$ is called the moment of momentum, or angular momentum of the particle with respect to O , i.e.,

$$\vec{H} = \vec{OP} \times \vec{p} = \vec{r} \times m \frac{d\vec{r}}{dt} = m \left(\vec{r} \times \frac{d\vec{r}}{dt} \right).$$

Newton's 2nd Law: The time rate of change of linear momentum of a particle is proportional to the applied/imposed force and takes place in the direction in which the force acts.

$$\vec{F} \propto \frac{d\vec{p}}{dt}$$

$$\vec{F} = km \frac{d\vec{v}}{dt} = km \frac{d^2 \vec{r}}{dt^2}$$

$$\vec{F} = m\vec{a}.$$

Example 9. (Motion under gravity) If a moving particle of mass m be subject to the action of gravity alone then the equation of motion of the particle is

$$m \frac{d^2 \vec{r}}{dt^2} = -mg\hat{k},$$

where $\hat{k}$ is the unit vector drawn vertically upwards.

$$\begin{aligned} \frac{d^2 \vec{r}}{dt^2} &= -g\hat{k} \\ \vec{r}(t) &= -\frac{g}{2}t^2\hat{k} + t\vec{e} + \vec{f}, \end{aligned}$$

where $t = 0$, $\vec{v} = \vec{u}_0$, and $\vec{r} = 0$, then $\vec{e} = \vec{u}_0$, and $\vec{f} = \vec{0}$. Hence, $\vec{r}(t) = -\frac{1}{2}gt^2\hat{k} + \vec{u}_0t$. The locus of $\vec{r}$ is a plane curve determined by the vectors $\vec{u}_0$ and $\hat{k}$.

Kinetic energy: If a particle of a mass m moves with a velocity $\vec{v}$ then the kinetic energy is given by

$$T = \frac{1}{2}m|\vec{v}|^2.$$

Potential energy: Let a particle of mass m be placed at height h with respect given frame of reference and let g be the gravity. Then the potential energy of the particle is given by

$$V = mgh.$$

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